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A lattice model related to the nonlinear Schroedinger equation

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This is a historical note. In 1981 we constructed a discrete version of quantum nonlinear Schroedinger equation. This led to our discovery of quantum determinant: it appeared in construction of anti-pod (11). Later these became important in quantum groups: it describes center of Yang-Baxter algebra. Our paper was published in Doklady Akademii Nauk vol 259, page 76 (July l981) in Russian language.

1. Inverse scattering method is used (in classical [1, 2] and quantum cases [3]) to solve evolutionary equations of completely integrable dynamical systems. In quantum case we shall abbreviate the method to qism. It is based on Lax representation

$$\partial_t L_n(\lambda) = M_{n+1}(\lambda)L_n(\lambda) - L_n(\lambda)M_n(\lambda) \tag{1}$$

Entries of the matrices $L_n(\lambda)$ and $M_n(\lambda)$ are expressed in terms of dynamical variables of the lattice system (they also depend on spectral parameter λ). The monodromy matrix $T^{n,m}(\lambda) = L_n(\lambda)\dots L_m(\lambda)$ $(n \geq m)$ and transfer matrix $\tau(\lambda) = \mathrm{tr} T^{N,1}(\lambda)$ are important for qism. Recently an effective method of construction of action-angle variables was discovered, it is based on R-matrix. In classical case [4] it provides Poisson brackets of matrix elements of the monodromy matrix, while in quantum case [3, 5] it gives commutation relations:

$$[T_c(\lambda) \otimes T_c(\mu)] = [T_c(\lambda) \bigotimes T_c(\mu), R_c(\lambda,\mu)] \tag{2}$$

$$R_q(\lambda,\mu)\left(T_q(\lambda) \bigotimes T_q(\mu)\right) = \left(I \bigotimes T_q(\mu)\right)\left(T_q(\lambda) \bigotimes I\right) R_q(\lambda,\mu) \tag{3}$$

These equations lead to commutativity of transfer matrices (we use subindex c or q to distinguish classical from quantum). Meaning that $\partial_\mu \ln \tau(\mu)$ is a generating functional of Hamiltonians for completely integrable systems.

In classical case E.K. Sklyanin proved [7] that corresponding equations of motion can be represented in the Lax form (1). Here we prove that this is true in quantum case as well. Generating functional of the operators $M_n(\lambda)$ is a matrix $m_n(\lambda,\mu)$:

$$m_n(\lambda,\mu) = i\tau^{-1}(\mu)\partial_\mu \tau(\mu) - iq_n^{-1}(\lambda,\mu)\partial_\mu q_n(\lambda,\mu)$$
$$q_n(\lambda,\mu) = \mathrm{tr}_2\left(I \bigotimes T^{N,n}(\mu)\right) R_q^{-1}(\lambda,\mu)\left(I \bigotimes T^{n-1,1}(\mu)\right)$$
$$i\left[\partial_\mu \ln \tau(\mu), L_n(\lambda)\right] = m_{n+1}(\lambda,\mu)L_n(\lambda) - L_n(\lambda)m_n(\lambda,\mu)$$

Here tr_2 denotes trace in the second linear space of the tensor product. The proof follows from $q_{n+1}(\lambda,\mu)L_n(\lambda) = L_n(\lambda)q_n(\lambda,\mu)$. So we proved that even in the quantum case it is sufficient to have $L_n(\lambda)$ operator and R matrix in order to apply QISM.

2. Let consider nonlinear Schroedinger equation (nS). In continuous case it has a Hamiltonian

$$H = \int dx\left(\partial_x\psi^\dagger\partial_x\psi + \kappa\psi^\dagger\psi^\dagger\psi\psi\right),$$
$$\{\psi_c(x), \psi_c^\dagger(y)\} = i\delta(x-y), \qquad \left[\psi_q(x), \psi_q^\dagger(y) = \delta(x-y)\right] \tag{4}$$

It is integrable both in classical [8] and quantum [3, 9] cases. Corresponding R- matrix can be called quasi-classical

$$R_q = I \bigotimes I - iR_c, \qquad R_c = \frac{\kappa\Pi}{\lambda-\mu} \tag{5}$$

Here I is identical matrix 2X2 and Π is parmutation matrix.

Lattice generalization of nS has long attracted attention of the experts [12, 13]. We propose a new version of lattice nS both in classical and quantum cases. It is distinguishing feature is that R-matrix is the same as in

the continuous case and basic variables χ are canonical Bose fields. We start by suggesting the following L_n operator:

$$
\begin{gathered}
L_n(\lambda) = -i\lambda\Delta\sigma_3/2 + S_n^3 I + S_n^+\sigma_+ + S_n^-\sigma_-, \\
S^3 = 1 + \tfrac{\kappa}{2}\chi_n^\dagger\chi_n, \qquad S_n^+ = -i\sqrt{\kappa}\chi_n^\dagger\rho_n^+, \qquad S_n^- = i\sqrt{\kappa}\rho_n^-\chi_n \\
\rho_n^\pm = \rho_n^\pm(\chi_n\chi_n^\dagger), \qquad \rho_n^+\rho_n^- = 1 + \tfrac{\kappa}{4}\chi_n^\dagger\chi_n, \qquad 2\sigma_\pm = \sigma_1 \pm i\sigma_2 \\
\{\chi_m^c, {\chi_n^c}^\dagger\} = i\Delta\delta_{m,n}, \qquad \left[\chi_m^q, {\chi_n^q}^\dagger\right] = \Delta\delta_{m,n}
\end{gathered} \tag{6}
$$

Here σ are Pauli matrices. We consider repulsive case $\kappa > 0$ and put $\rho_n^+ = \rho_n^- = \rho_n$.

3. Here we shall discuss lattice model (6) in classical case. Simple calculations lead to

$$
\begin{gathered}
T(\lambda)\sigma_2 T^t(\lambda)\sigma_2 = d_c^{n-m+1}(\lambda) I \\
d_c(\lambda) = \det L_n(\lambda) = 1 + \lambda^2\Delta^2/4
\end{gathered} \tag{7}
$$

This shows that at $\lambda = \nu = -2i/\Delta$ the $L_n(\lambda)$ operator turns into one dimensional projector. This makes it possible to calculate explicitly logarithmic derivatives of $\tau(\lambda)$ at this point, which can be represented as a sum of local densities:

$$
\begin{gathered}
\partial_\mu^n \ln\tau(\lambda)|_{\lambda=\nu} = \textstyle\sum_{k=1}^N h_{k,n}, \\
h_{k,n} = D^n \ln \mathrm{tr} L_{k+n}(\nu) L_{k+n-1}(\lambda_{k+n-1}) \ldots L_k(\lambda_k) L_{k-1}(\nu)|_{\lambda_{.}=\nu}
\end{gathered} \tag{8}
$$

Here D^n is a differential operator. For small n it is:

$$
\begin{gathered}
D^1 = \partial_k = \tfrac{d}{d\lambda_k}, \qquad D^2 = 2\partial_{k+1}\partial_k + \partial_k^2 \\
D^3 = 6\partial_{k+2}\partial_{k+1}\partial_k + 6\partial_{k+2}^2\partial_{k+1} + 6\partial_{k+2}\partial_{k+1}^2 - 6\partial_{k+2}^2\partial_k - 6\partial_{k+2}\partial_k^2 + \partial_k^3
\end{gathered} \tag{9}
$$

We use this notations to define lattice classical Hamiltonian of nS

$$
\begin{gathered}
H_c = D_c(\lambda)\ln\left[(1+\lambda/\nu)^{-N}\,\tau(\lambda)\right] + \text{complex conjugate} \\
D_c(\lambda) = \tfrac{i}{12\kappa}\left(\tfrac{d}{d\lambda^{-1}}\right)^3
\end{gathered} \tag{10}
$$

The explicit expression shows that this Hamiltonian describes interaction of five nearest neighbors on the lattice. In the continuous limit [$\chi_n = \psi_n\Delta$; $\psi_{n+1} - \psi_n = O(\Delta)$; $\Delta \to 0$, $N \to \infty$ but $N\Delta = \text{const}$] it goes to the correct Hamiltonian of the continuous model (4) and L_n operator (6) turns into correct continuous L_n operator, see [8].

4. Here we construct quantum lattice nS model. Quantum analog of (7) is given by

$$
\begin{gathered}
T(\lambda)\sigma_2 T^t(\lambda + i\kappa)\sigma_2 = d_q^{n-m+1}(\lambda) I \\
d_q(\lambda) = \Delta^2(\lambda - \nu)(\lambda - \nu + i\kappa)/4
\end{gathered} \tag{11}
$$

This defines **quantum determinant**:

$$
\det_q T(\lambda) = T_{11}(\lambda)T_{22}(\lambda + i\kappa) - T_{12}(\lambda)T_{21}(\lambda + i\kappa) = d_q^{n-m+1}(\lambda)
$$

To define Hamiltonian of the model let us add quantum correction like in [3]:

$$
H_q = \left(D_c(\lambda) + \frac{i\kappa}{6}\frac{d}{d\lambda^{-1}}\right)\ln\left[(1+\lambda/\nu)^{-N}\,\tau(\lambda)\right] + \text{hermitian conjugate} \tag{12}
$$

The model can be solved by qism [3]. The pseudo-vacuum Ω is annihilated by lattice Bose fields $\chi_n\Omega = 0$. The eigenvectors are given by algebraic Bethe ansatz:

$$
\Psi(\lambda_1 \ldots \lambda_n) = B(\lambda_1)\ldots B(\lambda_n)\Omega, \qquad B(\lambda) = T_{12}(\lambda)
$$

These λ_j satisfy a system of Bethe equations:

$$
\left(\frac{1 - i\lambda_j\Delta/2}{1 + i\lambda_j\Delta/2}\right)^N = \prod_{k\neq j}\frac{\lambda_j - \lambda_k - i\kappa}{\lambda_j - \lambda_k + i\kappa} \tag{13}
$$

Corresponding eigenvalue of $\tau(\lambda)$ is

$$\left(1-\frac{i\lambda\Delta}{2}\right)^N \prod_{k=1}^{n} \frac{\lambda-\lambda_k+i\kappa}{\lambda-\lambda_k} + \left(1+\frac{i\lambda\Delta}{2}\right)^N \prod_{k=1}^{n} \frac{\lambda_k-\lambda+i\kappa}{\lambda_k-\lambda} \tag{14}$$

From here we obtain energy levels [eigenvalues of the Hamiltonian]

$$H_q\Psi = \left(\sum_{k=1}^{n} E(\lambda_k)\right)\Psi, \qquad E(\mu) = f(\mu) + \overline{f(\overline{\mu})}$$
$$f(\mu) = \left(D_c + \tfrac{i\kappa}{6}\tfrac{d}{d\lambda^{-1}}\right)\ln\left(\tfrac{\mu-\lambda+i\kappa}{\mu-\lambda}\right)|_{\lambda=\nu}$$

The Hamiltonian has correct continuous limit (4) and $E(\mu) \to \mu^2$.

5. Quantum nS model constructed above can be considered as a generalization of XXX model with negative spin $-2/\kappa\Delta$. We can rewrite the L_n operator (6) in the way similar to XXX:

$$L_n^X = -\sigma_3 L_n = i\lambda + t_n^k \otimes \sigma_k$$

Here t_n^k are simple linear combinations of S_n^k from (6). They form an infinite dimensional representation of $SU(2)$ algebra, see [14].

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